International Forum
Vol. 13, No. 2
October 2010
pp. 83-92

FEATURE

Improving Students' Mathematical Thinking

Samuel Gaikwad

Abstract: *Mathematical thinking is understood and appreciated in academic circles. Such thinking differs from thinking in other subjects in its vocabulary, symbols, and grammar. It requires working with multiple solutions and problem generation by thinking across concepts and thinking about thinking in a more complex way. This article suggests ways that teachers can help students achieve their math potential, including teaching them in more enjoyable ways, and by addressing them through their preferred learning styles.*

"I try so very hard—why don't all my students get it?" "I show many examples in class yet, in general, why aren't they interested in learning math?" These questions haunt math teachers around the world. "Math is boring. I don't know why we have to learn these equations" are the standard student concerns. Where does the problem lie? Is it the fault of the students or the teachers? I suggest that it is neither. The fault is in the content delivery process. One may ask, "Isn't delivery of content basically the same for all content areas? Or are you suggesting thinking is different for different types of content?" Does one think differently when one studies English than when studying science, mathematics, or social sciences?

Mathematical thinking, I believe, does differ significantly from thinking in other subjects, in certain, specific ways. This paper discusses some of these differences, and suggests ways to improve both the thinking and the enjoyment of the students while learning about math. I have been teaching mathematics for over 35 years at the secondary, tertiary, and graduate levels. Presented herein are some of the techniques I have used successfully to help my students develop mathematical thinking through experimentation. A few references are provided in the paper to support my ideas on improving students' mathematical thinking, and to show how other math teachers, to a great degree, think the same way.

Making Math Enjoyable

Mathematical thinking requires effort on the part of the thinker. If a student as a thinker does not find items to think about that are worthwhile, which to some degree depends on the interest of the student, then he/she may not put in the effort required to think. Students are put off by monotony, and numbers can be monotonous. Creating interest in the subject, then, becomes the teacher's responsibility. Fortunately, it is possible to make math interesting, even likable. People like to hear stories, and students are no different. Stories associated with mathematicians and numbers, and delivered with a moral punch line laced with humor can do wonders in math learning.

Humor

It is not easy for children to grasp mathematical concepts, because, even though teachers say they are giving concrete examples, numbers are altogether abstract. Just take a simple example: It is difficult for children to comprehend the symbol *2* as representing two of something and then when 3 of the same thing are added to it, it mysteriously becomes 5—a very different symbol—an abstract concept. It is almost magical. To lighten up the mental strain, and to keep the focus on the work at hand, I suggest teachers use some topic-appropriate humor as they go along in their teaching. Teachers can easily delve into the reservoir of their personal experiential anecdotes, or those of others. They can select a comic strip or a cartoon quip. The possibilities are numerous. When a teacher surprises students with humor occasionally, they will learn to look forward to math class.

Games, Riddles, and Puzzles

Mathematical games can be very engaging. Appropriate games can teach the desired concept with uncanny ease, and at the same time, make learning fun. I have used manipulatives (for example, dice in a statistics class to teach probability) in teaching to even greater advantage: to build group cohesiveness, accountability, and shared responsibility. Felling and Currah (2007) provide interesting math games for children, and many more materials are available on the internet or in bookstores.

Riddles and puzzles are inherent attention catchers. Teachers can create topic related riddles and puzzles or get them from other sources. These help children to be sleuths, analyzing the problem, synthesizing the probable answer(s), and then judging the worth of the solution. This may even lead them to generate extensions to the original question.

Build the Base with the Basics

Unless the foundation is strong, the structure of a building is going to be weak. So it is with mathematics, especially because math concepts are layered; and layers, for the most part, follow a definite order. The foundation for math is the ability to read and understand mathematical language. Mathematical and numerical skills build on these often-overlooked basic language and literacy skills which are not directly related to numbers, but are a required part of the foundation.

Reading Mathematical Language

Math uses language not found in everyday conversation; hence it requires a conscious effort on the part of the teachers and the students to use it correctly. Every field has its own vocabulary and ways of thinking and documenting ideas. These ways vary across disciplines, but often the rules for communicating in the discipline are not explained clearly to the students. It is to the teacher's advantage to adequately and meaningfully familiarize students with math vocabulary, symbols, and meaning of the 'topic sentence' (Kenney, Hancewicz, Heuer, Metsisto, & Tuttle, 2005), so that students can focus on math skills with confidence, having mastered the "language" of the field.

Vocabulary

Math vocabulary is unique to its content. For example: *similar* means *alike* in the English language, but in math it means that the *ratios of the corresponding sides are equivalent and the angles are equal*. *Odd* means *different* or *strange* in regular use, but in math it means numbers that are *not even*. Students have to become familiar with the words, their meaning, and use. Practice with the way the words are used in math is paramount.

Symbols

Symbols make up the shorthand of mathematics. Math texts are short and dense, and contain more concepts per sentence than any other type of text; yet they are written in fewer words and symbols.

Math text cannot be read quickly. The text usually contains words, mixed with numeric and/or non-numeric symbols. The page layout may not be left to right, but may have a different pattern. The text may contain graphs, tables, or figures of different kinds. Considering all this, the teacher needs to help students read the information carefully, ascribing definite meaning to each symbol.

Topic Sentence

In the English language, the topic sentence is generally placed at the beginning of a paragraph and all that follows is in support of that initial sentence. However, in mathematics the key idea frequently comes at the end of the paragraph, as a question or a statement. It is important read all the way to the end carefully to know what is asked for before selecting the relevant data and planning a strategy for the solution(s).

Visualizing the Concepts: Making the Abstract Concrete

If students can see the concrete facets of an abstract concept, it may facilitate learning (De Paul Center for Urban Education, 2007). Using figures of speech to achieve this is one way to proceed. Helping students visualize the concept using literary constructs is one way to help make the idea more concrete. Let me share a few examples of how this could be achieved:

Simile: Would you say a graph is like a road on a map? Why?
Metaphor: Why would one say infinity is a bottomless pit?
Analogy: How is addition like a water pond? (see also Brunsting & Moirao, 2007)

There are no absolute answers to questions like these, but it gets students to be active thinkers and share ideas, which frequently may surprise teachers. When a student is able to visualize a concept, he/she will be able to connect it meaningfully and creatively with other known concepts to further visualize a broader picture, and see additional connections.

Creative Ways to Improve Mathematical Thinking

Mathematical skills are essentially taught using math concepts. Usually, the teacher tries to develop the desired skills in students by presenting structured procedures and multiple practice exercises. This is good and it is needed, but this system may restrict the students' development of independent reasoning and intuitive problem solving abilities. If students are to learn a broader set of skills and concepts, math cannot be taught simply as a recipe to be followed.

Finding Solutions in Multiple Ways

Teachers need to help students look at mathematical problem in multiple ways. This section contains concepts about teaching mathematical reasoning and creativity that were suggested by Benson et al. (2005), for which I have developed examples and discussion.

Multiple solutions. Math teachers commonly teach a unidirectional approach to problem solving, which leads to a single solution. Once the desired answer is obtained, the inquiry ends and the thinking stops. This is an age-old practice that has worked to create efficient and swift computers, and people who use well-tested techniques to reach a known goal. This sort of mathematical thinking is a necessary, but not sufficient process for building efficient, critical, and creative thinkers.

When students are told they are to think creatively in offering alternatives to the obvious answers, the following task could be offered. For example, say the teacher asks the students to give the answer for 2 + 3 = ____. If they say 5, the teacher may say it is correct, but not acceptable at the moment. But if they answer 4 + 1, or 8 – 3, the teacher will say the answer is correct and acceptable. Students may be encouraged to use a combination of mathematical operations to arrive at the desired result. This process teaches students multiple ways of looking at a problem, and makes it more likely that students will come to the same answer using different thinking processes. This process works for higher math, too.

Solutions leading to problem generation. Students need to learn to see beyond the obvious. Straightforward solutions or recipe-like problem-solving routines are the typical ways students are taught. But the challenge lies in application, stretching the limits, and in achieving transferability of the concepts learned. Here are some ways to attain these objectives.

Placing restrictions. If the problem is about the area of triangles, restrict it to only isosceles triangles, then encourage students to generate problems involving principles learned and apply the same to other forms of triangles and other polygons. If the problem is about trigonometry, restrict it to sine and cosine in order to understand and generate other trigonometric functions.

Relaxing the conditions. If the problem is about fractions, find out how things change when the fraction is written in decimal or percentage form. If the study is about quadratic equations, see how the procedure changes for obtaining solutions with matrices. Here the student is not restricted to using only one mode of obtaining solution.

Altering the details. If the problem is about addition and subtraction, see how the procedures change when substituting with multiplication and division. If the problem deals with rectangles, see how the conditions change when the figure is converted to a parallelogram.

Checking for uniqueness. Ask students directly if something can be done in more than one way. Or if something can be done in several ways, ask for the most convenient or most unique way.

Thinking About Thinking

Students will be able to solve complex math problems if they learn to think about their thinking process and understand how they actually go about finding a solution. One of the ways to accomplish this is to write out the thought process used in solving a given problem. Once they process their thinking and put it down in their own words, students will be able to solve other problems involving a similar process or a combination of several thought processes. Teachers can help students develop this skill.

As students solve math problems, ask them to explain in their own words the procedures, skills, and knowledge used in arriving at the solution. This process helps them think reflectively and to understand more clearly how they think. Kenney et al. (2005) argue that knowing one's own thinking process assists students in generating other similar problems of their own, and in solving them by checking the logic and consistency of the concepts at hand. This eventually could help students stretch their thinking skills and to develop new and more complex problem solving abilities. To achieve this, the teacher may even ask students to write down in prose form every step they used in their thinking process to arrive at the answer, such as, 'I started out with . . . , because . . . . Then I did . . . ,' and so on.

This activity may be unwelcome due to its emphasis on writing, which is difficult for some students. Students could, alternatively, explain their processes to each other in pairs or small groups orally, which is usually perceived as more fun than writing, or the teacher could ask selected students to explain their process from the front, and the others could compare their own processes with those of others. In any case, metacognition—understanding one's own mental process in problem solving—is an important part of math learning and should be emphasized and practiced.

Math Circle

One way to enhance mathematical thinking among students is to promote "Math Circle" in the school. This is an out-of-class exercise organized by the school or math teachers to develop math aptitude in interested students in order to take them to a higher level of mathematical thinking and to introduce more complex problem solving. Vandervelde (2009) gives detailed information on how to start and maintain such a "circle."

The success of the "Math Circle" depends on the commitment by the school administrators, teachers, parents, and students. It involves 2-3-hour sessions weekly or fortnightly led by interested math professionals who will take students further in their imagination about and understanding of mathematical concepts. Professionals are chosen based on their expertise. Such programs have been the practice for decades in Russia, Bulgaria, and other countries (Vandervelde, 2009).

In summary, problem solving is a creative process. The process demands multiple solutions as well as finding answers in multiple ways and then deciding on the most attractive solution. Thinking about the process of arriving at a solution is a complex mental activity that may lead to transfer of learning. 'Math circle' is one way to stretch students' imagination and help them think outside the box. To facilitate mathematical thinking, reaching students through their preferred learning style would be desirable. The following section deals with this possibility.

Teaching to Math Learning Styles

Carl Jung created a model of learning styles, viz., perception function (sensing and intuition), and judgment function (thinking and feeling). Brunsting and Moirao (2009) adapted Jung's learning style model to create a mathematical learning style model with four categories. They combined sensing and thinking to form Mastery, sensing and feeling to form Interpersonal, feeling and intuition to form Self-Expressive, and intuition and thinking to form Understanding (see Figure 1).

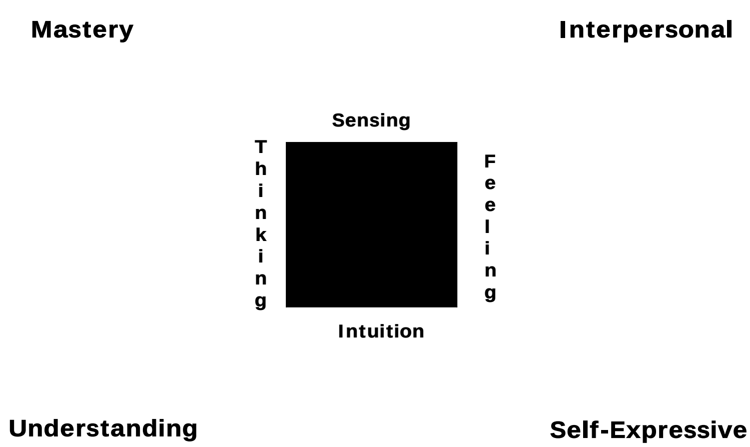

Figure 1. *Model of mathematical learning styles.*

Note. From *Making Mathematics Authentic, Relevant, and Rigorous for Every Student* (p. 14), by J. Brunsting and D. Moirao. Paper presented at ASCD Annual Conference, Anaheim, California, March, 2007.

Based on the model presented in Figure 1, research was conducted (n = 2000) to better understand what kind of math learners are most prevalent, and whether math learning styles are associated with differences in mathematical ability (Brunsting & Moirao, 2009). Results showed that 35% of the learners fell into the Mastery Learners category and of these, 12% failed in math. Only 10% were in Understanding category, and only 1% of these failed in math. A total of 20% of those tested were in Self-Expressive group, and 22% of these failed in math. Another 35% of learners were classified as belonging to the Interpersonal group, and of these, 65% failed in math (Brunsting & Moirao, 2009).

This result reveals cause for concern, but also possibilities for helping students more effectively, as we better understand what causes them to do poorly in math. Each one of these learning style groups has special skills with possible overlap. Teachers need to have knowledge and expertise in these skills to reach the students and meet their learning needs. Teachers could become knowledgeable in these skills by attending staff development workshops related to specific aspects of learning styles or by taking undergraduate or graduate courses specific to this area. Table 1 provides some suggestions, summarized from Brunsting and Moirao's (2007; 2009) presentations, as to how teachers could help students learn through their dominant learning style.

Conclusion

Mathematical thinking is not just a process to get to the solution, but a way of working to stretch the edges of the problem and study it in depth without changing the principles, and to develop the skills to generate new problems. Mathematical thinking involves knowing and practicing in an engaging way the delivery of math concepts. It starts by building the mathematical base through learning to read mathematical language and visualize the concepts. Further, it requires working with multiple solutions and problem generation by thinking across links. One way to achieve this is through forming a "Math Circle" in the school.

To facilitate mathematical thinking, it is suggested that the teacher provide students with opportunities to learn through their more dominant or preferred learning styles. Content may be approached in the ways suggested in this article, designed to fit the learning styles of the students, and presented in interesting ways. These kinds of activities will help students improve their mathematical thinking ability, and it will help them reach their math potential.

Table 1

Suggestions for Teaching to Specific Math Learning Styles

Mastery Learners	**Interpersonal Learners**
• Tell students your expectations and give clear directions to achieve them • Provide practice for skill development with appropriate feedback • Use visual aids such as concrete objects, figures, diagrams, etc. • Use manipulatives • Help students develop mental map of the problems • Show students relationship among topics	• Pick math problems from day to day life experiences • Help children 'experience' the math situation under consideration • Get students to work in groups • Create a conducive environment for sharing solutions • Allow students to teach each other • Provide practice with feedback
Understanding Learners	**Self-Expressive Learners**
• Provide reasons for doing the work they are doing • Ask students to write their thinking process • Encourage mental math over the use of algorithms • Encourage students to guess answers and then test them for accuracy • Move from simple to complex problems sooner, rather than later	• Help students imagine the mathematical concept • Engage students in research • Do not emphasize computational accuracy as the only goal of mathematics • Provide students opportunities to see math in art forms • Encourage students to talk about their encounters with math problems • Have more math solution workouts be mental, rather than on paper

References

Benson, S., Addington, S., Archavsky, N., Cuoco, A., Goldenberg, E. P., & Karrnowaski, E. (2005). *Ways to think about mathematics*. Thousand Oaks, CA: Corwin Press.

Brunsting, J., & Moirao, D. R. (2007). *Making mathematics authentic, relevant, and rigorous for every student*. Paper presented at ASCD Annual Conference, Anaheim, California, March, 17-19, 2007.

Brunsting, J., & Moirao, D. R. (2009). *Mathematical styles and strategies for differentiating instruction and increasing student engagement*. Paper presented at ASCD Annual Conference, Orlando, Florida, March 13-16, 2009.

De Paul Center for Urban Education (2007). *Make math verbal and visual*. Paper presented at the ASCD Annual Conference, Anaheim, California, March 17-19, 2007.

Felling, J., & Currah, J. (2007). *What's your game plan?* Paper presented at the ASCD Annual Conference, Anaheim, California, March 17-19, 2007.

Kenney, J. M., Hancewicz, E., Heuer, L., Metsisto, D., & Tuttle, C. L. (2005). *Literary strategies for improving mathematics instruction*. Alexandria, VA: Association for Supervision and Curriculum Development.

Vandervelde, S. (2009). *Circle in a box*. Berkeley, CA: Mathematical Science Research Institute.

Samuel Gaikwad, PhD
Professor, Education Department
Adventist International Institute of Advanced Studies
Silang, Cavite, Philippines